The Birational Transformations of Algebraic Curves of Genus Four.

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In this paper the groups of birational transformations which leave curves of genus 4 invariant are obtained, and some geometrical properties connected with such transformations are considered. This has been done for the space sextic situated upon an hyperboloid by Wiman,* who has also outlined in the same paper the groups for plane curves derived from those sextics which lie upon a cone.

The discussion naturally falls into three main divisions: first, the hyperboloidal case, which deals with binodal quintics and a few sextics which are the projections of a space sextic lying upon a hyperboloid; second, the conical case, wherein the corresponding normal curve is on a cone; and, third, the hyperelliptic case.

§ 1. *The Hyperboloidal Case.*

1. As is well known, the normal form for every plane curve of genus 4 which is not hyperelliptic is a curve of the fifth order. That is, all such curves $C_m(4)$, whatever be the degree m, possess a point-group series g_5^2 and are thus birationally equivalent to a $C_5(4)$, a quintic with two nodes.

When the nodes are distinct, the triangle of reference OIJ can be selected with vertices I and J or (0, 0, 1) and (1, 0, 0) at the nodes. Then the triply infinite system of adjoint conics is written

$$axy + by^2 + cyz + dxz = 0.$$

Whence putting for xy, y^2, yz, and xz the new variables $\rho x'$, $\rho y'$, $\rho z'$, and $\rho w'$, the result is a plane section of the hyperboloid F_2 whose equation is

$$x'z' = y'w'.$$

* A. Wiman: Ueber die algebraischen Curven von den Geschlechtern $p=4$, 5, und 6, welche eindeutige Transformationen in sich besitzen, *Bihang till Kongl. Svenska Vetenkaps-Akademiens Handlingar.* Stockholm, 1895–96.

The substitution in the quintic gives a cubic surface F_3 whose intersection with F_2 is a space sextic S_6.

Whenever, by a birational transformation, the plane quintic C_5 goes into itself, the corresponding space sextic S_6 must also go into itself, and conversely. The adjoint conics ϕ_2 merely interchange, as do also the corresponding plane sections of F_2. Hence the transformations in space are all linear. Since $p=4$, every generator of F_2 is a trisecant. The inflexional tangents of S_6 must be generators of F_2, for no other straight lines have three points in common with the curve. S_6 can have no point-singularity; for if projected from a double point or cusp it would give a quartic whose genus is less than four.

Collineations which leave S_6 invariant, leave F_2 invariant. The linear transformations of F_2 are of two kinds: first, those in which the systems of generators interchange; second, those in which the systems do not interchange, though the generators of either or of both systems may interchange among themselves. It is then convenient to have coördinates that distinguish between the systems of generators. This is done by the following equations:

$$x'/w' = y'/z' = y_1/y_2 \text{ and } y'/x' = z'/w' = x_1/x_2,$$

which represent two pairs of planes intersecting respectively on F_2 in lines which intersect and thus belong to different systems. Then x_1, x_2 and y_1, y_2 may be regarded as coördinates of the two systems.

A cubic in both sets x_1/x_2, y_1/y_2, viz.,

$$\theta_3(x_1/x_2,\ y_1/y_2) = 0,$$

will then determine three generators of one system for any particular generator of the other, and therefore define a sextic curve whose trisecants are generators of the hyperboloid.

Such a sextic, if projected* from a point on the curve, goes into a plane quintic with two distinct nodes; from a point on F_2 but not on the curve, a sextic with two triple points; from a point not on F_2, a sextic with six nodes lying on a conic, unless the center of projection be taken at the vertex of a cubic cone on which S_6 lies, in which case the plane curve is manifestly a cubic counted twice.

*Clebsch: Vorlesungen über Geometrie II, S. 414.

If the center of projection be taken at the vertex D or (0, 0, 0, 1) of the reference tetraedron, the substitutions are

$$\begin{pmatrix} x', & y', & z', & w' \\ xy, & y^2, & yz, & xz \end{pmatrix} \text{ or } \begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y, & x, & y, & z \end{pmatrix},$$

where the variables in the lower row are to replace those in the upper.

Collineations of the first kind wherein the generator systems interchange may be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ ay_1 + by_2, & cy_1 + dy_2, & ex_1 + fx_2, & gx_1 + hx_2 \end{pmatrix},$$

which, if generators through two invariant points, say $x_1 = y_1 = 0$ and $x_2 = y_2 = 0$, be taken as edges of the reference tetraedron, may be reduced to the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ b_1 y_1, & b_2 y_2, & a_1 x_1, & a_2 x_2 \end{pmatrix}.$$

Collineations of the second kind may leave the generators of one system severally invariant while interchanging those of the other system, or they may interchange generators of both systems among themselves. Such are

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ ax_2, & bx_1, & y_1, & y_2 \end{pmatrix} \text{ and } \begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ ax_2, & bx_1, & cy_2, & dy_1 \end{pmatrix}.$$

The product of two collineations of the first kind is evidently either identity or one of the second kind. When therefore an equation is invariant under an operation T of the first kind, it is likewise invariant under T^2, which is an operation of the second kind if the period of T exceeds two. Thus the group which contains operations interchanging the systems of generators contains as many which leave the systems invariant. When the systems do not interchange, the groups of collineations are simply isomorphic with the well-known binary linear groups. Hence there are no new simple groups.

2. If the S_6 can be transformed into itself by a collineation of period 2, there are two cases arising according as the transformation is of the first or of the second kind. First suppose that the curve belongs to a group G_2 of order 2 under which the generator systems interchange. The collineation may be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y_1, & y_2, & x_1, & x_2 \end{pmatrix},$$

or, in space coördinates,

$$\begin{pmatrix} x', & y', & z', & w' \\ z', & y', & x', & w' \end{pmatrix},$$

which, when projected from the vertex (0, 0, 0, 1), becomes, for the plane,

$$\begin{pmatrix} x, & y, & z \\ z, & y, & x \end{pmatrix}.$$

Hereafter accents will be omitted from the space coördinates.

It is evident that the space collineation is a central perspective whose center $x+z=y=w=0$ is the pole as to F_2 of the plane of perspective $x-z=0$.

The reference tetraedron can be so chosen that two vertices D or $x_1=y_1=0$ and B or $x_2=y_2=0$ are on the curve. Then $a_0=d_3=0$ in the equation

$$\begin{aligned} &x_2^3y_2^3\theta_3(x_1/x_2,\ y_1/y_2) \\ &\equiv x_1^3\sum_{n=0}^{3} a_n y_1^{3-n}y_2^n + x_1^2x_2\sum_{n=0}^{3} b_n y_1^{3-n}y_2^n + x_1x_2^2\sum_{n=0}^{3} c_n y_1^{3-n}y_2^n + x_2^3\sum_{n=0}^{3} d_n y_1^{3-n}y_2^n = 0. \qquad \text{(I)}\end{aligned}$$

In order that this equation be invariant under the above transformation, the following conditions must be fulfilled:

$$a_1=b_0,\ a_2=c_0,\ a_3=d_0,\ b_2=c_1,\ b_3=d_1,\ c_3=d_2.$$

Then (I) reduces to

$$\begin{aligned} &a_3(x_1^3y_2^3+x_2^3y_1^3)+(a_2x_1y_1+b_3x_2y_2)(x_1^2y_2^2+x_2^2y_1^2) \\ &\quad +(a_1x_1^2y_1^2+b_2x_1x_2y_1y_2+c_3x_2^2y_2^2)(x_1y_2+x_2y_1)+x_1x_2y_1y_2(b_1x_1y_1+c_3x_2y_2)=0,\end{aligned}$$

or, in space coördinates,

$$a_3(x^3+z^3)+(a_2y+b_3w)(x^2+z^2)+(a_1y^2+b_2yw+c_3w^2)(x+z)+yw(b_1y+c_3w)=0.$$

This cubic surface and F_2 intersect in the sextic curve. It projects from D into the plane quintic

$$\begin{aligned} &a_3y^2(x^3+z^3)+y(a_2y^2+b_3xz)(x^2+z^2)+(a_1y^4+b_2xy^2z+c_3x^2z^2)(x+z) \\ &\qquad\qquad +xyz(b_1y^2+c_3xz)=0.\end{aligned}$$

The first two of these coefficients can be reduced to unity by change of scale. The quintic is then written

$$\begin{aligned} &y^2(x^3+z^3)+y(y^2+axz)(x^2+z^2)+(by^4+cxy^2z+dx^2z^2)(x+z) \\ &\qquad\qquad +xyz(ey^2+fyz)=0, \qquad (1)\end{aligned}$$

with the transformation

$$H\equiv\begin{pmatrix} x, & y, & z \\ z, & y, & x \end{pmatrix},$$

which transformation is unaltered by the above change of scale.

3. As was observed before, the space collineation for (1) is a central involution with center V at $(1, 0, -1, 0)$. The center is not on the hyperboloid and every line through it cuts the hyperboloid twice. Then all lines to points

of S_6 from V are bisecants, generators of a cubic cone K_3. The cone is clearly of order 3, for any plane through V meets S_6 in six points which must lie in pairs on three bisecants. The points P_i ($i = 1, 2, \ldots, 6$) in the plane of perspective $x - z = 0$ go into themselves by the transformation of period 2, hence the bisecants VP_i are tangent to S_6 at P_i. Two of the points P_i are at the vertices (0, 1, 0, 0) and (0, 0, 0, 1) of the reference tetraedron). The cubic cone has nine inflexional generators which lie by threes in twelve planes. An inflexional tangent plane to K_3 contains but two distinct points of S_6. These are interchanged by the transformation of period 2. Hence such a plane is a double-osculating plane. When VP_i is an inflexional generator, the corresponding inflexional tangent plane is a sextactic plane. In any case an osculating plane at P_i contains at least four consecutive points of S_6. For the osculating plane contains three consecutive points of S_6, hence two consecutive bisecants, on each of which there is a pair of consecutive points. Such a plane is therefore stationary.

The quintic is derived from the sextic by projection from the vertex D of the reference tetraedron $ABCD$. Let lines from D to A, B, C, V pierce the plane of projection in J, O, I, D' respectively. Then D', the image of D, is on the quintic and is the center of the plane homology H. A tangent at D' must have either three- or five-point contact. The five lines from D' to the points in which C_5 intersects the axis of homology are tangent to the curve at those points. The nodes at I and J are interchanged by H and are both of the same kind, crunodes, acnodes or cusps according as a^2-4d is positive, negative or zero. Unless they are cusps, the class of the curve is 16. So the five ordinary tangents from D' and the inflexional tangent at D' leave in general four double tangents from D'. In the case when the tangent at D' has contact of the fourth order, there will be but three double tangents. The double tangents are the images of the planes tangent to K_3 which intersect K_3 in the generator VD. There will be four such planes except when VD is an inflexional generator. If, however, one of them is tangent also to F_2, the plane curve is bicuspidal and the bitangents from D' are images of the other three tangent planes.

The nine inflexional tangent planes to K_3 give as images nine adjoint conics each having double three-point contact with C_5. The images of the eighteen points of intersection of F_2 with the nine inflexional generators of K_3 lie by sixes on twelve adjoint conics, because the cross-sections of K_3 are non-singular cubics whose points of inflexion lie by threes on twelve lines, and a plane containing

two inflexional generators must therefore contain a third. So an adjoint conic containing four of the eighteen three-point contact points must contain six.

Moreover, since the osculating planes of S_6 at the six points in the plane of perspective have four-point contact, their images, adjoint conics, will have four-point contact with C_5. The image, however, of the osculating plane at D breaks up into the y-axis and a line through D' tangent to the curve. As the y-axis intersects C_5 in D', the tangent at D' must have three-point contact. This is another proof for the inflexion at D'. When the osculating plane at D has six-point contact, its image, a line through D', will of course have five-point contact.

Only four bitangents and one inflexional tangent are in general accounted for. The rest, sixty-four and thirty-two respectively, meet in pairs on the axis $x - z = 0$.

Any plane through V meets S_6 in six points collinear in pairs with V. Such a plane section of F_2 goes into an adjoint conic ϕ_2 which cuts out a point-group, G_6, the points of which lie in pairs on three lines through D'.

If the projection had been made from A or C instead of D, a sextic would have been obtained remaining invariant under a quadratic inversion.

4. The collineation of period 2 of the second kind may be put in the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & -x_2, & -y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & -y, & z, & -w \end{pmatrix}$$

which projects from D into the homology

$$T \equiv \begin{pmatrix} x, & y, & z \\ x, & -y, & z \end{pmatrix}.$$

Beginning as before with equation (I), it is clear that in all terms the sum of the exponents of x_2 and y_1 must be odd or even in order that the above transformation may leave the equation unaltered. Thus either

$$a_0 = a_2 = b_1 = b_3 = c_0 = c_2 = d_1 = d_3 = 0,$$

or

$$a_1 = a_3 = b_0 = b_2 = c_1 = c_3 = d_0 = d_2 = 0.$$

Either set of conditions gives essentially the same quintic. Selecting the former set, the equation is

$$a_1 x_1^3 y_1^2 y_2 + a_3 x_1^3 y_2^3 + b_0 x_1^2 x_2 y_1^3 + b_2 x_1^2 x_2 y_1 y_2^2 + c_1 x_1 x_2^2 y_1^2 y_2 + c_3 x_1 x_2^2 y_2^3 + d_0 x_2^3 y_1^3 + d_2 x_2^3 y_1 y_2^2 = 0,$$

or, in space coördinates,

$$d_0 x^3 + a_3 z^3 + xz(c_1 x + b_2 z) + x(b_0 y^2 + d_2 w^2) + z(a_1 y^2 + c_3 w^2) = 0.$$

By change of scale of x, y and z the coefficients b_0, d_0 and d_2 can be absorbed and the equation be written

$$x^3 + xz(ax + bz) + cz^3 + z(dw^2 + ey^2) + x(y^2 + w^2) = 0,$$

which gives, for the plane quintic,

$$x^3(y^2 + z^2) + xy^2z(ax + bz) + z^3(cy^2 + dx^2) + y^4(x + ez) = 0. \tag{2}$$

The collineation in space is an axial involution. That is, under it $y = w = 0$ and $x = z = 0$ are not only invariant as a whole but all their points are invariant, because any plane through either line goes into itself. Hence a line joining two corresponding points of the S_6 must intersect these two axes and S_6 lies on a ruled surface with the two invariant lines as directrices. The directrix $y = w = 0$ does not intersect S_6, but any plane through it cuts S_6 in six points which lie on three bisecants through the point P, where the second directrix $x = z = 0$ cuts the above plane. Thus $x = z = 0$ is a triple directrix. Similarly $y = w = 0$ is a double directrix, for $x = z = 0$ contains two points of S_6, leaving four points in any plane through it to lie in pairs on two bisecants which meet in a point on $y = w = 0$. So the curve lies on an R_5, the order 5 being the sum of the multiplicities of the straight line directrices. From a point on a bisecant there are five other bisecants, which generate a ruled surface on which the given bisecant is of order 5. Any plane containing a bisecant cuts S_6 in four points besides the two on the bisecant. These four points are joined by six bisecants. Therefore the complete plane section of the surface is these six lines and the given bisecant which counts as five. The order of the ruled surface is thus 11. As R_{11} has a factor R_5, there is also an R_6.

It is important to bear in mind that the curve belonging to a G_2 of the first kind lies on a cubic cone, while the one whose group G_2 is of the second kind is on a particular ruled surface of order 5. Under special conditions the ruled surface of order 6 may break up, as will presently appear. The curve lies on a K_3 when it has a central involution, on an R_5 when the involution is axial.

5. The quintic (2) in the form obtained has at least one acnode. The center of the plane perspective is at O or $(0, 1, 0)$, which point is an inflexion. The inflexional tangent is $x + ez = 0$. This has contact of the fourth order when $ae^2 + c = e^3 + be$. The quintic cuts the y-axis in a point D', the image of D on S_6, and the tangent to C_5 at D' is $x + dz = 0$. The curve has six double tangents from O, which, with the tangent at D' and the inflexional tangent at O, make up the number 16, the class of the curve.

When $d=e$, in equation (2), the curve

$$x^3(y^2+z^2)+xy^2z(ax+bz)+z^3(cy^2+dx^2)+y^4(dz+x)=0 \tag{3}$$

possesses, in addition to the linear transformation T, the quadric transformation

$$Q\equiv\begin{pmatrix}x, & y, & z\\ xy, & xz, & yz\end{pmatrix},$$

and thus QT, which is also quadric. The two quadric transformations correspond to two central collineations in space; and T, the product of the two quadric transformations, corresponds, as we have seen, to an axial involution. The central collineations project into quadric transformations because the center of projection is not an invariant point of either collineation.

If the center of projection were taken at A, the result instead of the quintic (2) would have been the sextic

$$y^3w^3+yz^2w(ayw+bz^2)+cz^6+z^4(dw^2+ey^2)+yz^2w(y^2+w^2)=0 \tag{2$'$}$$ *

whose transformation

$$T'\equiv\begin{pmatrix}y, & z, & w\\ y, & -z, & w\end{pmatrix}$$

corresponds to the above T, and to the axial space collineation. Moreover, when $d=e$ the sextic (3$'$) is obtained with collineations Q' and $Q'T'$ or

$$\begin{pmatrix}y, & z, & w\\ w, & \pm z, & y\end{pmatrix},$$

which correspond to the central space collineations.

The sextic (2$'$) has six bitangents from A', the point on IJ which is the image of A, and the sextic (3$'$) has in addition to the six from A', nine bitangents from O.

The condition $d=e$ for (3) might have been obtained in two ways: either by working for the S_6 which allows interchange of the systems of generators as well as the collineation for (2), or by finding the condition that the R_6 for the space sextic of equation (2) breaks up into two K_3. The latter method is as follows: the plane BCD or $x=0$ of the reference tetraedron contains two trisecants BC and CD which are cut by S_6 in $(0, 1, 0, 0)$, $(0, \sqrt{c}, \pm i\sqrt{e}, 0)$ and $(0, 0, 0, 1)$, $(0, 0, \pm i\sqrt{d}, \sqrt{c})$ respectively. Likewise $z=0$ contains the points $(1, 0, 0, \pm i)$ and $(1\pm i, 0, 0)$. If the two sides of the quadrilateral formed by bisecants in the x-plane which meet on $x=z=0$ are met at the same

* In this section, the plane sextics are given accented numbers, and corresponding quintics unaccented.

point by those of the quadrilateral in the z-plane, R_6 breaks up. The condition that a bisecant in the x-plane meets a corresponding one in the z-plane is that certain four of the above points lie in the same plane, which condition is satisfied only if

$$\begin{vmatrix} 1, & i, & 0, & 0 \\ 1, & 0, & 0, & i \\ 0, & 0, & i\sqrt{d}, & \sqrt{c} \\ 0, & \sqrt{c}, & i\sqrt{e}, & 0 \end{vmatrix} = 0;$$

that is, if $\sqrt{c}(\sqrt{d} - \sqrt{e}) = 0$. When $c = 0$, four points in the plane $x = 0$ coincide at C and there would be no determinant. When, however, $d = e$, there are two central involutions in addition to the axial involution of (2), and S_6 is on two K_3 into which R_6 has broken up. When $c = 0$, it is observed that although there is no new transformation the curve has a stationary plane at C.

The centers V_1, V_2, or $(0, 1, 0, \pm 1)$, are on the line $x = z = 0$ and they project from D into O of the reference triangle OIJ. The plane $x + dz = 0$ is a double osculating plane at $(0, 0, 0, 1)$ and $(0, 1, 0, 0)$ and is an inflexional tangent plane to both cones. Its image is a double tangent with contact of the second order at O and ordinary contact on the z-axis. This inflexional double tangent and the six ordinary double tangents make up the number 16, the class of the curve. The eight other inflexional tangent planes to the two cones have as images adjoint conics, each having double contact of the second order with C_5.

6. A four-group can also be obtained whose operations are all of the second kind or axial. Besides the collineation belonging to (2) there may be two others of the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_2, & \pm x_1, & \pm y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ z, & \pm w, & x, & \pm y \end{pmatrix},$$

or, for the plane,

$$\begin{pmatrix} x, & y, & z \\ yz, & \pm xz, & xy \end{pmatrix}.$$

Beginning with the form for (2) before the coefficients were absorbed by change of scale, and obtaining the condition that these last transformations will leave the equation invariant, the result is

$$x^3 + z^3 + axz(x + z) + b(xy^2 + zw^2) + c(xw^2 + y^2z) = 0,$$

which gives the plane quintic

$$y^2(x^3 + z^3) + axy^2z(x + z) + y^4(bx + cz) + x^2z^2(bz + cx) = 0. \qquad (4)$$

The lines $x \pm z = 0$ and the adjoint conics $xz \pm y^2 = 0$ are invariant under all the plane transformations of the group. They are the projections of sections of F_2 made by invariant planes $y \pm w = 0$ and $x \pm z = 0$. Each of the invariant conics intersects C_5 in six points collinear in pairs with O. The conics $xy \pm z^2 = 0$ and $yz \pm x^2 = 0$ are each invariant under the corresponding quadric transformations of the group. They are the projections of the intersections of F_2 with the cones $xy \pm z^2 = 0$ and $yz \pm x^2 = 0$. The cone $xy + z^2 = 0$ is transformed by $\begin{pmatrix} x, & y, & z, & w \\ z, & w, & x, & y \end{pmatrix}$ into $zw + x^2 = 0$. If from the last equation and that of F_2, w be eliminated, the result is $xy + z^2 = 0$. The cone $xy + z^2 = 0$ has the line $y = z = 0$ in common with the hyperboloid, and it goes into $x = w = 0$ by the above transformation. The cone and the hyperboloid have also a cubic curve in common which remains invariant under the above transformation and is projected from D into the conic invariant under the corresponding plane transformation.

The quadric transformations for C_5 may be regarded as the products of inversions by harmonic homologies. Thus:

$$\begin{pmatrix} x, & y, & z \\ yz, & xz, & xy \end{pmatrix} = \begin{pmatrix} x, & y, & z \\ xy, & xz, & yz \end{pmatrix}\begin{pmatrix} x, & y, & z \\ z, & y, & x \end{pmatrix} = \begin{pmatrix} x, & y, & z \\ xz, & yz, & xy \end{pmatrix}\begin{pmatrix} x, & y, & z \\ y, & x, & z \end{pmatrix}, \text{ etc.}$$

The quintic (4) has an inflexion at O, but the tangent from O to the point of intersection of C_5 with the y-axis is not, as in (3), the inflexional tangent at O, but is transformed into it by the quadric transformations. The former tangent is $cx + bz = 0$, while its image, the inflexional tangent, is $bx + cz = 0$.

7. There are then two non-cyclic groups of order 4. The question arises whether there can be a cyclic G_4. In a collineation of period 4 the systems of generators must interchange, for otherwise there would be fixed generators on which the points of S_6 would have to change cyclically, which could only be when the period is 3; or else the twelve points on the edges of the reference tetraedron must be situated in groups at the vertices, which latter case occurs, as will be seen, presently in the collineation of period 5.

A collineation of period 4 can be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ iy_1, & y_2, & x_1, & -ix_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ z, & iy, & x, & -iw \end{pmatrix},$$

or, in the plane,

$$\begin{pmatrix} x, & y, & z \\ z, & iy, & x \end{pmatrix}.$$

As the square of this collineation is the G_2 of the second equation, it is well to begin with the form for that equation before the coefficients were absorbed. With the conditions that the above transformation imposes, it is found that

$$x_1^3y_2^3 + x_2^3y_1^3 + ax_1x_2y_1y_2(x_1y_2 + x_2y_1) + (bx_1^2y_1^2 + cx_2^2y_2^2)(x_1y_2 - x_2y_1) = 0,$$

which possesses not only the above transformation but also

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \sqrt{c}x_2, & \sqrt{-b}x_1, & \sqrt{c}y_2, & \sqrt{-b}y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \sqrt{-bc}z, & cw, & \sqrt{-bc}x, & by \end{pmatrix},$$

which becomes, in the plane,

$$\begin{pmatrix} x, & y, & z \\ \sqrt{-b}yz, & \sqrt{c}xz, & \sqrt{-b}xy \end{pmatrix}.$$

From the above equation is derived the plane quintic

$$x^3y^2 + y^2z^3 + axy^2z(x+z) + (by^4 + cx^2z^2)(z-x) = 0, \tag{5}$$

and the transformations form a diedral G_8, which contains a four-group of each kind as well as the cyclic G_4.

The quintic has the geometrical properties of equation (3). The inflexional tangent $x - z = 0$ has also ordinary contact at $(1, 0, 1)$. From O there are six bitangents, and one of the points I or J is an acnode.

These six groups, composed of two G_2, three G_4, and one G_8, are the only ones without higher prime periods; so we pass now to the consideration of the transformations of period 3.

8. A cyclic G_3 can not interchange the systems of generators; for evidently, in interchange of systems, the period of the transformation must be even. There are then two kinds of G_6 to be considered: one which leaves neither system invariant, and one which leaves one system invariant. The first kind may be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & \omega x_2, & \omega^2 y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \omega^2 y, & z, & \omega w \end{pmatrix},$$

and, in the plane,

$$\begin{pmatrix} x, & y, & z \\ x, & \omega y, & z \end{pmatrix},$$

where $\omega^3 = 1$.

The substitution is made in (I) as before, and the conditions for invariance obtained. There are three resulting equations, but two are discarded because they are of lower genus. The third is

$$x_1^3y_2^3 + x_2^3y_1^3 + a(x_1^3y_1^3 + x_2^3y_2^3) + bx_1x_2^2y_1^2y_2 + cx_1^2x_2y_1y_2^2 = 0,$$

or, in space coördinates,

$$x^3 + z^3 + a(y^3 + w^3) + xz(bx + cz) = 0.$$

This projects into the sextic

$$y^3(x^3 + z^3) + a(y^6 + x^3z^3) + xy^3z(bx + cz) = 0. \tag{6'}$$

It is evident, on account of the symmetry of the equation, that the systems of generators can also interchange, and the equation possesses the following transformations of period 2:

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y_2, & y_1, & \omega x_2, & \omega^2 x_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \omega w, & z, & \omega^2 y \end{pmatrix},$$

or, in the plane,

$$\begin{pmatrix} x, & y, & z \\ xy, & \omega xz, & yz \end{pmatrix}.$$

The centers for these space collineations are $(0, 1, 0, -\omega)$, and the planes of perspective are $y - \omega w = 0$.

Equation (6) belongs therefore to a diedral G_6. The planes $x + kz = 0$ are invariant under all transformations of the group. The centers of the three cubic cones of perspective are on the line $x = z = 0$. By the transformations of period 3 the cones are interchanged. Therefore a plane $x + kz = 0$ which is tangent to one cone is tangent to all three. Through any point on the line connecting the vertices there are six lines tangent to a section made by a plane which contains the point but not the line, for such a section is a non-singular cubic and is therefore of class 6. There are thus six planes through BD each tangent to one cubic cone and hence tangent to all. A plane through BD tangent to the cones touches the sextic in three points, one in each plane of perspective, and at each of the eighteen such points of tangency there is a stationary plane. The twenty-seven inflexional tangent planes must each touch S_6 in but two points.

Corresponding to the above properties of S_6, the plane sextic has six tritangents from O, which accords with the class of the curve 18. Each of the points of tangency determines an adjoint conic with four-point contact. The points of tangency, like those in (1), lie by sixes on twelve adjoint conics. There are twenty-seven adjoint conics with double three-point contact.

The last S_6 does not pass through a vertex of the reference tetraedron, so that the projection from any vertex is a sextic with two triple points. A binodal quintic is obtained by inversion of the sextic, the triangle of inversion

being selected with all the vertices O, I, and J on the sextic, I and J being at the triple points. This gives the quintic

$$ay^5 - 3a^{1/3}y^4z + 3a^{2/3}y^3z^2 + x^3y^2 + (1-a^2)y^2z^3 + bx^2y^2z + cxy^2z^2 \\ - 3a^{1/3}x^3yz - 2a^{1/3}bx^2yz^2 - a^{1/3}cxyz^3 + 3a^{2/3}x^3z^2 + a^{2/3}bx^2z^3 = 0, \quad (6)$$

which is transformed into itself by

$$\begin{pmatrix} x, & y, & z \\ \omega x\{\omega^2 y + a^{1/3}(1-\omega^2)z\}, & y\{\omega^2 y + a^{1/3}(1-\omega^2)z\}, & yz \end{pmatrix}$$

and by

$$\begin{pmatrix} v, & y, & z \\ vy, & vz, & yz \end{pmatrix},$$

where $v \equiv \omega x + a^{1/3}y$.

The same quintic may be obtained by transforming S_6 to a new reference tetraedron $A'BCD'$ whose vertex D' is at $(a^{1/3}, 0, 0, -1)$ on S_6 and whose reference planes are $x' = x + a^{1/3}w$ and $y' = y + a^{1/3}z$. The projection into a quintic is then made from the new vertex D'. The transformations from the original tetraedron to the plane are

$$\begin{pmatrix} x, & y, & z, & w \\ xy - a^{1/3}xz, & y^2 - a^{1/3}yz, & yz, & xz \end{pmatrix}.$$

The planes $x + kz = 0$, which as before stated are invariant under all operations of the group, go into the pencil of conics $xy - a^{1/3}xz + kyz = 0$ through OIJ. So the six points cut out on C_5 by any conic of the pencil can only interchange among themselves. Six conics of the pencil are triply tangent to C_5, since six planes of the corresponding axial pencil are triply tangent to S_6. They correspond also to the six tritangents from O in the plane sextic. Through each of the eighteen points of tangency there are likewise adjoint conics which have four-point contact with C_5.

The above transformations of period 2 are clearly inversions, with v, y, z as sides of the triangle of inversion. There are three centers O_1, O_2 and O_3, corresponding to the three values of ω in the line $v \equiv \omega x + a^{1/3}y = 0$. The coördinates of these centers referred to the OIJ triangle are $(a^{1/3}, -\omega, 0)$, corresponding to the coördinates $(a^{1/3}, -\omega, 0, 1)$ of the vertices of the cubic cones referred to the tetraedron $A'BCD'$. The three centers are points of intersection of $z = 0$ with C_5. They are not, however, inflexions; hence there are fourteen tangents to C_5 from each center besides the one at the center.

If S_6 be projected from A or C, the transformations will be all linear of the form

$$\begin{pmatrix} y, & z, & w \\ \omega z, & \omega^2 y, & w \end{pmatrix} \text{ and } \begin{pmatrix} y, & z, & w \\ \omega y, & \omega^2 z, & w \end{pmatrix};$$

and the plane sextic is

$$y^3w^3 + z^6 + az^3(y^3 + w^3) + yz^2w(byw + cz^2) = 0. \tag{6''}$$

9. Consider all groups whose operations do not interchange the systems of generators, which contain the above G_3 as an invariant subgroup. Such a group could not be a cyclic G_6, for there are not six points of S_6 to permute cyclically on an invariant generator. It can not be an octaedron group, for that contains a cyclic G_4 which is excluded on the same ground as the G_6. It can not be a diedral G_{12}, for that contains a cyclic G_6. There remain only the diedral G_6 and the tetraedral group.

It is evident that the space sextic $(6')$, when $b = c$, is invariant also under

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_2, & \omega x_1, & \omega^2 y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ z, & \omega^2 w, & x, & \omega y \end{pmatrix},$$

or

$$\begin{pmatrix} x, & y, & z \\ yz, & \omega xz, & xy \end{pmatrix}$$

for the plane sextic projected from D.

The second sextic $(6'')$, when $b = c$, has the quadric transformation

$$\begin{pmatrix} y, & z, & w \\ \omega^2 zw, & yw, & \omega yz \end{pmatrix}.$$

The above transformations, compounded with those for the previous diedral G_6, give transformations of periods 3 and 6, making up a diedral G_{12}, one-half of whose operations interchange the systems of generators.

The quintic (6), when $b = c$, becomes

$$\begin{aligned} ay^5 - 3a^{1/3}y^4z + 3a^{2/3}y^3z^2 + x^3y^2 + (1 - a^2)y^2z^3 + bx^2y^2z + bxy^2z^2 \\ - 3a^{1/3}x^3yz - 2a^{1/3}bx^2yz^2 - a^{1/3}bxyz^3 + 3a^{2/3}x^3z^2 + a^{2/3}bx^2z^3 = 0; \end{aligned} \tag{7}$$

and it has in addition to the transformations of the diedral G_6 the following:

$$\begin{pmatrix} x, & y, & z \\ y\{\omega^2 z + a^{1/3}(y - a^{1/3}z)\}, & x\{\omega^2 z + a^{1/3}(y - a^{1/3}z)\}, & x(y - a^{1/3}z) \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ z(y + a^{1/3}\omega^2 x), & (\omega y + a^{1/3}x)(y - a^{1/3}z), & x(y - a^{1/3}z) \end{pmatrix}.$$

10. Assuming that the subgroup which does not change the generating systems is a tetraedron group, since this group contains an invariant axial four-group, it is convenient to start with the equation of the S_6 for (4) and find what further conditions exist among the coefficients to admit the collineation

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \pm i(x_1 \pm x_2), & (x_1 \mp x_2), & (y_1 \mp y_2), & \mp i(y_1 \pm y_2) \end{pmatrix}.$$

These conditions are $a = 3$ and $b = -c$, and the resulting equation is

$$(x_1y_2 + x_2y_1)^3 + b(x_1y_1 - x_2y_2)(x_1y_1 + x_2y_2)(x_2y_1 - x_1y_2) = 0,$$

or

$$(x + z)^3 + b(y - w)(y + w)(x - z) = 0,$$

from which is obtained

$$y^2(x + z)^3 + b(y^2 - xz)(y^2 + xz)(x - z) = 0. \tag{8}$$

These equations are invariant not only under all the substitutions of the tetraedron group, but also under the remaining substitutions of the octaedron group, so that in addition to the substitutions for (4) the quintic has

$$\begin{pmatrix} x, & y, & z \\ \pm xz + xy \pm y^2 - yz, & (\pm i)^\epsilon(\pm xz + xy \mp y^2 - yz), & \mp xz + xy \mp y^2 + yz \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ (x \mp iy)(y \pm iz), & (i)^\epsilon(x \pm iy)(y \pm iz), & (x \pm iy)(y \mp iz) \end{pmatrix},$$

where $\epsilon = 1, 2$.

Under all operations of the octaedron group, the plane $x + z = 0$ and its pole P or $(1, 0, 1, 0)$ with respect to F_2 are invariant. Hence the set of six bisecants from P is invariant. There are but six bisecants from P, for if there were an infinite number, there would be a central involution

$$\begin{pmatrix} x, & y, & z, & w \\ x, & -w, & z, & -y \end{pmatrix},$$

and this substitution does not leave (8) invariant. The tangents to S_6 at the points in the invariant plane are invariant as a whole under the transformations of the group. Hence they must either lie in the invariant plane or pass through the invariant point. They can not lie in the invariant plane, for in that case S_6 would have twelve points in the plane; so they must pass through P. Moreover, the six points on S_6 in the invariant plane are in six-fold involution.

11. The octaedron group G_{24} has six central and three axial involutions, and the vertices of the six cubic cones on which the space sextic lies are in the

invariant plane. They are the vertices of a complete quadrilateral whose diagonal triangle, together with the point P, determines the tetraedron whose three pairs of edges are axes of the axial involutions. The planes of perspective pass through P and cut the plane $x+z=0$ in six lines, which are the sides of a complete quadrangle whose diagonal triangle is the same as that for the complete quadrilateral. The diagonal triangle cuts F_2 in the six points on S_6. The set of planes and lines from P to the lines and points just described in the invariant plane, is invariant as a whole under the operations of the group.

Since the six planes of perspective and three axes of involution all go through P, it is evident that if P be taken as center* for the projection of S_6, a plane sextic will be obtained all of whose transformations of period 2 are linear. Moreover, as the other operations of G_{24} are generated by those of period 2, all the transformations in the group must be linear. Choose the invariant plane as plane of projection and the diagonal triangle as reference triangle. Then the equation of the F_2 section may be written

$$x^2+y^2+z^2=0,$$

and the points of S_6 in the plane are at $(0, \pm i, 1)$, $(1, 0, \pm i)$ and $(\pm i, 1, 0)$. At these points C_6 has cusps.

The above conic is invariant under six central homologies; viz.,

$$\begin{pmatrix} x, & y, & z \\ x, & z, & y \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ z, & y, & x \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ y, & x, & z \end{pmatrix},$$

$$\begin{pmatrix} x, & y, & z \\ -x, & z, & y \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ z, & -y, & x \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ y, & x, & -z \end{pmatrix};$$

and also under the following homologies, which are the projections of axial involutions in space:

$$\begin{pmatrix} x, & y, & z \\ -x, & y, & z \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ x, & -y, & z \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ x, & y, & -z \end{pmatrix}.$$

It is evident that a sextic invariant under the above transformations is symmetric in the variables and contains no odd powers. It has the form

$$a(x^6+y^6+z^6)+b(x^4y^2+x^2y^4+x^4z^2+x^2z^4+y^4z^2+y^2z^4)+cx^2y^2z^2=0.$$

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* The preceding equations (1)–(7) can likewise be projected from the invariant point P, and each is thus transformed into a sextic with six cusps lying on a conic.

By transferring the origin to one of the points at which there is a cusp, it is found that $b = 3a$ for genus 4.* So the equation can be written

$$(x^2 + y^2 + z^2)^3 + kx^2y^2z^2 = 0. \tag{8'}$$

This form shows that the axes are bicuspidal tangents.

The six-cuspidal sextic is of class 12, with 27 bitangents and 24 inflexions. From each center of the first six homologies there can be but six ordinary tangents to the points where the corresponding axes a_i cut C_6. Hence, from each center there are three double tangents, which means eighteen besides the cuspidal double tangents, each of which should be counted as three. This accounts for the entire number. The inflexional tangents meet in pairs on each of the six axes.

The binodal quintic (8) has, like (4), a point of inflexion at O, from which there are six bitangents. The inflexional tangent is $x - z = 0$, and it is also tangent to C_5 at the point of intersection with $y = 0$. The projection of P from D, or $(1, 0, 1)$, and the line $x + z = 0$ are invariant under all operations of the group. The six vertices of the complete quadrilateral project into points on $y + w = 0$. They are centers of quadric transformations of the group.

12. The G_3 already considered left neither system of generators invariant. Assume next that one system, say the y-system, is unchanged. Then G_3 can be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \omega x_1, & x_2, & y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \omega y, & \omega z, & w \end{pmatrix}.$$

This substitution in the general form (I) gives three equations, two of genus less than 4 and the other of the following form:

$$x_1^3 f_3(y_1, y_2) + x_2^3 \phi_3(y_1, y_2) = 0,$$

an equation containing eight constants. Of these eight, two can be absorbed in x_1 and x_2 respectively, and by a linear transformation

$$y_1 = ay_1' + by_2', \quad y_2 = cy_1' + dy_2'$$

the roots of f_3 can be prescribed. This leaves three moduli.

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* For certain values of the remaining modulus more double points may appear. Thus, for $k = -27$, $p = 0$. It is projectively equivalent to the astroid or four-cusp hypocycloid.

Instead, however, of reducing the number of arbitrary constants, the symmetry will be better preserved by writing the equation in the following form:

$$x_1^3(y_1 + a_1y_2)(y_1 + b_1y_2)(y_1 + c_1y_2) + x_2^3(y_1 + a_2y_2)(y_1 + b_2y_2)(y_1 + c_2y_2) = 0,$$

or, in space coördinates,

$$(y + a_1z)(y + b_1z)(y + c_1z) + (x + a_2w)(x + b_2w)(x + c_2w) = 0.$$

Projected from D, the result is the plane sextic

$$y^3(y + a_1z)(y + b_1z)(y + c_1z) + x^3(y + a_2z)(y + b_2z)(y + c_2z) = 0, \tag{9'}$$

with the transformations

$$\begin{pmatrix} x, & y, & z \\ \omega x, & y, & z \end{pmatrix}$$

into itself. The sextic has triple points at $x = y = 0$ and $y = z = 0$. The center of homology is at the latter point, and the axis $x = 0$ crosses the curve in three points of inflexion whose tangents pass through the center. Tangents at the center can have no residual points; hence they must be inflexional. Three inflexional tangents at the triple point, together with the three from the triple point to the points on the axis, make up 18, the class of the curve. There remain thirty inflexional tangents which meet by threes on the x-axis, the thirty points of inflexion lying by threes on ten lines through the center.

Since S_6 does not pass through a vertex of the reference tetraedron, it is convenient in order to determine the quintic (9) to select a new tetraedron whose vertex D' is at $(1, 0, 0, -a_2)$. If for x and y the variables $x' - a_2w$ and $y' - a_2z$ be substituted, the equation referred to the tetraedron $ABC'D'$ is

$$\{y' + (a_1 - a_2)z\}\{y' + (b_1 - a_2)z\}\{y' + (c_1 - a_2)z\}$$
$$+ x'\{x' + (b_2 - a_2)w\}\{x' + (c_2 - a_2)w\} = 0.$$

Projected from the vertex D', this gives the quintic

$$y^2\{y + (a_1 - a_2)z\}\{y + (b_1 - a_2)z\}\{y + (c_1 - a_2)z\}$$
$$+ x^3\{y + (b_2 - a_2)z\}\{y + (c_2 - a_2)z\} = 0, \tag{9}$$

which is transformed into itself by the same transformation as the last sextic.

This quintic has a double point at I, a cusp at J, and does not pass through O. The axis $y = 0$ is the cuspidal tangent. The axis $z = 0$ intersects the quintic in three points of inflexion. The tangents at these points pass through I, the center of the collineation. The cuspidal tangent and three inflexional tangents from the node count as seven, leaving eight to be accounted for by the tangents at

the node. In this case the inflexional tangents at points on the z-axis can not be again tangent at the node. It is therefore necessary that the nodal tangents have four-point contact.

No two of the constants can be equal, for if a_2 is equal to b_2 or c_2 the quintic breaks up. If a_2 is equal to a_1, b_1, or c_1, there is a triple point which reduces the genus. By symmetry it is evident that no two of the coefficients can be equal.

There are twenty-four inflexional tangents and fifty-four bitangents meeting in sets of threes on the z-axis. The points of inflexion lie by threes on eight lines through the center.

13. In order that the last S_6 may go into itself by any operation which changes the y_1 and y_2, the functions f_3 and ϕ_3 must either go into themselves or interchange. In the latter case x_1 and x_2 interchange. In the first case the Hessians H_f and H_ϕ must go into themselves. Assume that the Hessians are different. Each defines a pair of points. One pair may be the basis points of a G_2 which interchanges the other pair, or both pairs may be interchanged by the G_2, in which case the basis points are given by the functional determinant of H_f and H_ϕ. Neither of these cases is possible, for in order that the three points of $f_3=0$ and those of $\phi_3=0$ be put into themselves by a G_2, one of each must be invariant and form basis points. So to obtain groups of higher orders with G_3 as subgroup, f_3 and ϕ_3 must be interchangeable, and the six factors of $f_3\phi_3$ can be arranged so as to be cyclically permuted in one or more cycles. Denoting the six factors by a_1, a_2, a_3, b_1, b_2, b_3, it is evident, since an a_i goes into b_j, that there is a possibility of a cycle of six elements or of three cycles of two. When, moreover, f_3 goes into ϕ_3, H_f goes into H_ϕ. The latter can be done only by collineations of period 4 or 2. Hence the collineation is of period 2.

If then the collineation is assumed to be of the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_2, & x_1, & y_1, & -y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ y, & x, & -w, & -z \end{pmatrix},$$

the equation of the last space sextic in its first factored form is invariant when $a_1=-a_2$, $b_1=-b_2$, $c_1=-c_2$. The equation, after change of scale, becomes

$$(y+z)(y+az)(y+bz)+(x-w)(x-aw)(x-bw)=0.$$

This projects from D into the sextic

$$y^3(y+z)(y+az)(y+bz)+x^3(y-z)(y-az)(y-bz)=0, \tag{10'}$$

which belongs to the diedral G_6 whose transformations of period 3 are the same as those of the preceding equation, and one of period 2 is

$$\begin{pmatrix} x, & y, & z \\ y^2, & xy, & -xz \end{pmatrix}.$$

This sextic has of course the geometrical properties of (9′).

To obtain the plane quintic, transfer the above S_6 to a new tetraedron of reference whose vertex D' is at $(1, 0, 0, 1)$, with $x' = x - w$ and $y' = y - z$. Then

$$\{y' + 2z\}\{y' + (a-1)z\}\{y' + (b-1)z\} + x'\{x' - (a-1)w\}\{x' - (b-1)w\} = 0$$

gives the quintic

$$y^2\{y + 2z\}\{y + (a-1)z\}\{y + (b-1)z\} + x^3\{y - (a-1)z\}\{y - (b-1)z\} = 0. \quad (10)$$

The transformations of this quintic in addition to those for the last are

$$\begin{pmatrix} x, & y, & z \\ xz, & (2x-y)z, & \omega(2x-y)y \end{pmatrix} \text{ or } \begin{pmatrix} y, & z, & v \\ vz, & \omega vy, & yz \end{pmatrix},$$

where $v = 2x - y$.

The curve has the same general form as (9). There are fifteen tangents from J, since the class is 15. The cuspidal tangent counts for three, leaving twelve ordinary, six inflexional, or three inflexional and three ordinary tangents from J.

It is unnecessary to consider the forms where f_3 and ϕ_3 are identical, because the genus is lower in that case.

14. Since the four points of the Hessians can be paired in two ways, f_3 and ϕ_3 are each invariant under a transformation other than identity which is the product of two others. Such is true of

$$x_1^3 y_2(a y_1^2 + y_2^2) + x_2^3 y_1(y_1^2 + a y_2^2) = 0.$$

Or, putting x_i for y_i and y_i for x_i,

$$x_2 y_1^3(a x_1^2 + x_2^2) + x_1 y_2^3(x_1^2 + a x_2^2) = 0,$$

which projects from D into the quintic

$$xy^2(ay^2 + x^2) + z^3(y^2 + ax^2) = 0. \quad (11)$$

The collineation group of the S_6 is the diedral G_{12} which is generated by

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ -x_1, & x_2, & y_1, & -\omega y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & -y, & \omega z, & -\omega w \end{pmatrix}$$

and

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_2, & \pm x_1, & \pm\omega y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \omega z, & \pm\omega w, & x, & \pm y \end{pmatrix}.$$

The corresponding transformations in the plane are

$$\begin{pmatrix} x, & y, & z \\ x, & -y, & \omega z \end{pmatrix} \text{ and } \begin{pmatrix} x, & y, & z \\ yz, & \pm xz, & \omega xy \end{pmatrix}.$$

The curve passes through O, at which point $x = 0$ is an inflexional tangent. The lines $x^2 + ay^2 = 0$ are also inflexional tangents.

15. Consider next the case where H_f and H_ϕ are identical. The equation can then be put into the form

$$x_1^3y_2^3 + x_1^3y_1^3 + x_2^3y_2^3 + a^3x_2^3y_1^3 = 0, \quad \text{or} \quad a^3x^3 + y^3 + z^3 + w^3 = 0.$$

This projects into the sextic

$$a^3x^3y^3 + y^6 + y^3z^3 + x^3z^3 = 0, \tag{12'}$$

with triple points at I and J, or $x = y = 0$ and $y = z = 0$, and with three points of inflexion on the z-axis, tangents at which pass through the triple point as before.

The curves belong to a G_{36} generated by the following transformations:

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & x_2, & \omega y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \omega x, & \omega y, & z, & w \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ x, & y, & \omega z \end{pmatrix}$$

for the plane curve; and

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & \omega x_2, & y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \omega x, & y, & z, & \omega w \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ \omega x, & y, & z \end{pmatrix}.$$

The combination of these two groups of order 3 gives a G_9, invariant in the entire group. Moreover x_1 and x_2, y_1 and y_2 can also interchange by the collineation

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ ax_2, & x_1, & y_2/a, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ z/a, & w, & ax, & y \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ yz/a, & xz, & axy \end{pmatrix}.$$

This, taken with the G_9, forms a G_{18} simply isomorphic with the transformation group of the general plane cubic into itself. This is a subgroup of the G_{36} in which the systems interchange by

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y_2, & y_1, & x_2, & x_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & w, & z, & y \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ xy, & xz, & yz \end{pmatrix}.$$

The lines $x = z = 0$ and $y = w = 0$ are invariant in this group. The vertices of six cubic cones of perspective lie three on each of the invariant lines.

The lines are interchanged by the substitution

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & -x_2, & y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ -w, & z, & y, & -x \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ -xz, & yz, & y^2 \end{pmatrix}.$$

This belongs to the curve when $a^3 = -1$; that is,

$$y^3 + z^3 + w^3 - x^3 = 0,$$

or, for the plane,

$$y^6 + y^3z^3 + x^3z^3 - x^3y^3 = 0. \tag{13'}$$

This curve has thirty-six more collineations, and belongs therefore to a G_{72}.

The G_{72} is a permutation group of six things arranged in pairs of triads, the pairs being interchangeable. The G_{72} possesses three invariant sub-groups G_{36}.

The last S_6 lies on six R_5, in addition to the nine R_5 on which the preceding S_6 lies.

To obtain the quintic for the preceding equation, a change of coördinates is made as before. The new vertex D' can be taken at any one of the three points $(0, 0, 1, -t)$ on the line DC, where $t^3 = 1$. Here t is used instead of ω to distinguish between the transformations. By putting $z' = z + tw$ and $y' = y + tx$, we obtain the equation

$$(a^3 - 1)x^3 + y'^3 - 3txy'^2 + 3t^2x^2y' + z'^3 - 3tz'^2w + 3t^2z'w^2 = 0,$$

which projects into the quintic

$$(a^3 - 1)x^3y^2 + y^5 - 3txy^4 + 3t^2x^2y^3 + y^2z^3 - 3txyz^3 + 3t^2x^2z^3 = 0. \tag{12}$$

The transformation group G_{36} for the quintic (12) is generated by the following:

$$\begin{pmatrix} x, & y, & z \\ x, & y, & \omega z \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ xy, & y\{\omega y + t(1-\omega)x\}, & z\{\omega y + t(1-\omega)x\} \end{pmatrix},$$

$$\begin{pmatrix} x, & y, & z, \\ z(y - tx), & z(ax + ty - t^2x), & ay(ax + ty - t^2x) \end{pmatrix},$$

and

$$\begin{pmatrix} x, & y, & z \\ xy, & y(z + ty), & zy - t(xy + y^2 - xz) \end{pmatrix}.$$

As in the case of the last two quintics, there is a node which is the center of collineation of period 3, and the axis crosses the curve in points of inflexion, tangents at which pass through the center. There is likewise a cusp whose tangent passes through the node.

The G_{72} has also the following transformation for the quintic:

$$\begin{pmatrix} x, & y, & z \\ xz, & z(2tx - y), & y(2tx - y) \end{pmatrix}.$$

16. It remains to consider the case in which the curve will be invariant under a G_5. By collineations of odd periods the systems of generators are never interchanged. On the four fixed generators three points can not be permuted cyclically in fives. The three points on each generator must be at the vertices, the curve touching one generator and intersecting the other at each vertex. The five collineations can be put in the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \theta x_1, & \theta^4 x_2, & \theta^2 y_1, & \theta^3 y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \theta^2 y, & \theta^3 z, & \theta w \end{pmatrix},$$

where $\theta^5 = 1$. The curve has the equation

$$x_1^3 y_1^2 y_2 + x_1^2 x_2 y_2^3 + x_1 x_2^2 y_1^3 + a^5 x_2^3 y_1 y_2^2 = 0, \text{ or } x^2 y + y^2 z + z^2 w + a^5 x w^2 = 0.$$

It has, in addition to the above, the five collineations

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ a^2\theta^3 x_2, & x_1/a, & \theta y_2, & y_1/a \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \theta z/a, & a^2\theta^4 w, & a\theta^3 x, & y/a^2 \end{pmatrix}.$$

The equation belongs therefore to the diedral G_{10}.

The corresponding quintic is

$$x^2 y^3 + y^4 z + x y z^3 + a^5 x^3 z^2 = 0, \tag{14}$$

and its transformations are

$$\begin{pmatrix} x, & y, & z \\ \theta^3 x, & y, & \theta z \end{pmatrix} \text{ and } \begin{pmatrix} x, & y, & z \\ \theta yz/a, & a^2\theta^4 xz, & a\theta^3 xy \end{pmatrix}.$$

From the equation it is seen that $z = 0$ is tangent to the quintic at (0, 1, 0) and likewise at (1, 0, 0), which latter point is also a cusp. The other node is at (0, 0, 1), at which point $x = 0$ is an inflexional tangent.

All transformations of the group leave the conics $a^{5/2} x^2 \pm yz = 0$ invariant. Each quadric transformation leaves invariant the four corresponding conics $a^2\theta^2 xz \pm y^2 = 0$ and $a^{1/2}\theta^{1/2} xy \pm z^2 = 0$. This makes six conics invariant under

each quadric transformation. They are projections of the partial intersections of F_2 with certain quadric cones. The straight line intersections, as in (4), are not invariant under the transformations.

From O there are either ten ordinary or five double tangents besides OI and OJ, while from I and J there are ten ordinary or five inflexional tangents.

It appears that there can be no cyclic group of higher order than 5. The G_{60} is a subgroup of the generalized icosaedron group or the symmetric G_{120} which remains to be discussed.

There are five cyclic G_4 whose generating operations may be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \theta y_1, & \theta^2 y_2, & -\theta^3 x_2, & x_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ -\theta w, & -x, & \theta^2 y, & \theta^3 z \end{pmatrix},$$

which becomes, for the plane,

$$\begin{pmatrix} x, & y, & z \\ \theta xz, & xy, & -\theta^2 y^2 \end{pmatrix},$$

which transformations will belong to (14) when $a^5 = -1$. The new quintic is then

$$x^2y^3 + y^4z + x^3z^2 - xyz^3 = 0, \tag{15}$$

which is one form of the Bring* Curve. It possesses the linear transformations into itself whose types as given by Gordan† for the G_{60} are

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ -x_2, & x_1, & -y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ -z, & w, & -x, & y \end{pmatrix},$$

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & \theta^3 x_2, & \theta y_1, & \theta^2 y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \theta^2 y, & \theta^3 z, & \theta w \end{pmatrix},$$

and

$$\Big(\overset{x_1,}{\{(\theta + \theta^4)x_1 + x_2\}/(\theta^2 - \theta^3)}, \ \overset{x_2,}{\{x_1 - (\theta + \theta^4)x_2\}/(\theta^2 - \theta^3)},$$
$$\overset{y_1,}{\{(\theta^2 + \theta^3)y_1 + y_2\}/(\theta^4 - \theta)}, \ \overset{y_2}{\{y_1 - (\theta^2+\theta^3)y_2\}/(\theta^4 - \theta)} \Big)$$
$$= \Big(\overset{x,}{x + (\theta^2 + \theta^3)y + z - (\theta + \theta^4)w}, \ \overset{y,}{(\theta^2 + \theta^3)x - y + (\theta + \theta^4)z + w},$$
$$\overset{z,}{x + (\theta + \theta^4)y + z - (\theta^2 + \theta^3)w}, \ \overset{w}{-(\theta + \theta^4)x + y - (\theta^2 + \theta^3)z - w} \Big).$$

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* Klein: "Vorlesungen über das Ikosaeder."

† Gordan: "Ueber die Auflösung der Gleichungen vom fünften Grade," *Math. Ann.*, Vol. XIII.

These collineations, taken with

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y_2, & -y_1, & x_1, & x_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ -y, & z, & w, & -x \end{pmatrix},$$

generate the G_{120}. The collineations for the plane quintic are

$$\begin{pmatrix} x, & y, & z \\ yz, & -xz, & xy \end{pmatrix}, \begin{pmatrix} x, & y, & z \\ \theta x, & \theta^2 y, & z \end{pmatrix},$$

$$\begin{pmatrix} x, & y, \\ \{(\theta^2+\theta^3)x+y\}\{(\theta+\theta^4)y+z\}, & \{x-(\theta^2+\theta^3)y\}\{(\theta+\theta^4)y+z\}, \end{pmatrix.$$

$$\begin{matrix} z \\ \{x-(\theta^2+\theta^3)y\}\{y-(\theta+\theta^4)z\} \end{pmatrix},$$

and

$$\begin{pmatrix} x, & y, & z \\ y^2, & yz, & -xz \end{pmatrix}.$$

17. The space sextic for (15) can be expressed in pentaedral coördinates as the intersection of

$$F_3 \equiv \sum_{i=1}^{5} x_i^3, \quad F_2 \equiv \sum_{i=1}^{5} x_i^2, \quad F_1 \equiv \sum_{i=1}^{5} x_i$$

in space of four dimensions. Since, however, the last equation is linear, the curve lies in three-dimensional space, and one of the coördinates can be eliminated by means of the linear equation. There are then five reference planes

$$x_1 = 0, \ x_2 = 0, \ \ldots, \ x_5 = -(x_1 + x_2 + x_3 + x_4) = 0.$$

The pentaedron of réference is thus a tetraedron plus a fifth plane which does not pass through a vertex of the tetraedron. Any plane is cut by the four others in a complete quadrilateral. Through each of the ten vertices of the pentaedron three planes pass, and on each of the ten edges there are three vertices. It is clear that the curve belongs to the symmetric G_{120}.

The collineations which permute two of the coördinate axes are central involutions. Thus

$$\begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_1, & x_2, & x_3, & x_5, & x_4 \end{pmatrix}$$

has the center at $(0, 0, 0, 1)$ of the reference tetraedron $x_1x_2x_3x_4$, and the plane of perspective

$$x_4 - x_5 = x_1 + x_2 + x_3 + 2x_4 = 0.$$

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The plane of perspective cuts $x_4 = 0$ in the invariant line $x_1 + x_2 + x_3 = 0$ on which lie the three invariant vertices $(1, -1, 0, 0)$, $(1, 0, -1, 0)$ and $(0, 1, -1, 0)$. The other six vertices of the pentaedron lie in pairs on lines through the center and are interchanged by the above collineation. There are ten central perspectivities, corresponding to the ten vertices of the pentaedron. Hence the curve lies on ten cubic cones. The equation of any one of these cubic cones is obtained by eliminating one coördinate from F_2 and F_3 by means of F_1 and combining the results to eliminate a second coördinate. The result is the square of a cubic in the remaining three coördinates.

Now the symmetric G_{120} or generalized icosaedron group has five octaedron subgroups, each obtained by permuting all the variables but one. Let $x_1 x_2 x_3 x_5$ be the set of variables. Then the quadrilateral in the x_4-plane is invariant as a whole, but its sides and vertices are interchanged by the collineations. The sides are

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_1 + x_2 + x_3 = 0.$$

The equation of the plane sextic (8′) invariant under the symmetric G_{24} or octaedron group was found to be of the form

$$(x_1^2 + x_2^2 + x_3^2)^3 + k x_1^2 x_2^2 x_3^2 = 0.$$

The question naturally arose as to what value of k would give a sextic invariant not only under the G_{24} but also under the G_{120}. The plane sextic (8′) was referred to the diagonal triangle of the invariant quadrilateral. This suggested that the equations of S_6 be referred to the tetraedron whose base is the diagonal triangle

$$x_1 + x_2 = 0, \quad x_2 + x_3 = 0, \quad x_3 + x_1 = 0,$$

and whose vertex is the pole $(1, 1, 1, -4)$ of $x_4 = 0$ with respect to F_2. Denoting the new coördinates by x, y, z, w, the transformation of coördinates may be written

$$T^{-1} \equiv \begin{pmatrix} x, & y, & z, & w \\ 2x_2 + 2x_3 + x_4, & 2x_1 + 2x_3 + x_4, & 2x_1 + 2x_2 + x_4, & x_4 \end{pmatrix},$$

or

$$T \equiv \begin{pmatrix} x_1, & x_2, & x_3, & x_4 \\ -x + y + z - w, & x - y + z - w, & x + y - z - w, & 4w \end{pmatrix}.$$

The equations of S_6 become

$$F_2' \equiv x^2 + y^2 + z^2 + 5w^2 = 0,$$
$$F_3' \equiv 5w^3 - w(x^2 + y^2 + z^2) - 2xyz = 0.$$

The last equation combines with the preceding and reduces to $5w^3 - xyz = 0$. Eliminating w, we obtain a sextic

$$(x^2 + y^2 + z^2)^3 + 5x^2y^2z^2 = 0, \tag{15'}$$

which may be regarded as a sextic cone containing S_6, or as the plane projection of S_6 from a vertex of the new reference tetraedron upon its opposite face.

18. One of the collineations of period 5 may be written

$$R \equiv \begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_5, & x_1, & x_2, & x_3, & x_4 \end{pmatrix} = \begin{pmatrix} x_1, & x_2, & x_3, & x_4 \\ -(x_1 + x_2 + x_3 + x_4), & x_1, & x_2, & x_3 \end{pmatrix}.$$

This operation transformed through T gives

$$R' \equiv T^{-1}RT \equiv \begin{pmatrix} x, & y, & z, & w \\ x+y+3z-5w, & x-3y-z-5w, & -3x+y-z-5w, & x+y-z-w \end{pmatrix}.$$

To project this operation upon the plane $w = 0$ from the opposite vertex of the reference tetraedron, we may substitute the value of w in terms of x, y, z obtained from F_2' and F_3'; that is, $w = -xyz/(x^2 + y^2 + z^2)$. Then R' for the plane is

$$\begin{pmatrix} x, & y, & z \\ (x + y + 3z)(x^2 + y^2 + z^2) - 5xyz, & (x - 3y - z)(x^2 + y^2 + z^2) - 5xyz, & (-3x + y - z)(x^2 + y^2 + z^2) - 5xyz \end{pmatrix}.$$

The basis points of the transformation are at $(0, 1, \pm i)$, $(1, 0, \pm i)$, $(1, \pm i, 0)$, which points are cusps on the sextic. To the point of intersection of two lines will correspond three points of intersection of the two corresponding cubics. Hence the transformation is birational for the curve and not for the whole plane. We have thus an example of a periodic Riemann transformation.

Others can be obtained of periods 2, 3, 4 and 5 by the same method. The following one is of period 2:

$$\begin{pmatrix} x, & y, & z \\ (3x - y - z)(x^2 + y^2 + z^2) + 5xyz, & (-x + 3y - z)(x^2 + y^2 + z^2) + 5xyz, & (-x - y + 3z)(x^2 + y^2 + z^2) + 5xyz \end{pmatrix},$$

which comes from

$$\begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_1, & x_2, & x_3, & x_5, & x_4 \end{pmatrix}.$$

The other nine of period 2 are linear. Thus

$$\begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_1, & x_2, & x_5, & x_4, & x_3 \end{pmatrix} \text{ corresponds to } \begin{pmatrix} x, & y, & z \\ y, & x, & -z \end{pmatrix}.$$

Twenty-three operations of the G_{120} are linear, and the remaining ninety-six are Riemann transformations differing from those given above only in the signs, and the position of the coefficient 3.

§ 2. *The Conical Case.*

19. When the space sextic lies upon a cone K_2 of the second order, the projection from a point A on the surface will be either a tacnodal quintic or a sextic with three branches which have a common tangent, according as A lies upon or without the curve.

Let the tetraedron of reference be $ABCD$, with DA and DB generators of K_2, and DAB the polar plane of DC with respect to K_2. Then the equation of K_2 is

$$xy - z^2 = 0.$$

When K_2 is projected from A, the transformation may be written

$$\begin{pmatrix} x, & y, & z, & w \\ x'^2, & y'^2, & x'y', & y'z' \end{pmatrix}.$$

The plane section

$$ax + by + cz + dw = 0$$

is thus transformed into the conic

$$ax'^2 + by'^2 + cx'y' + dy'z' = 0,$$

which is tangent to $y' = 0$ at $x' = 0$. The vertices OIJ of the reference triangle are at the intersections of AB, AD, and AC, with the plane of projection. For simplicity in writing, the accents will be dropped from the plane coördinates.

The pencil of planes through AD defines the system of generators of K_2. They project into a pencil of lines $x = ky$, each line of which cuts the curve in three points besides those at I, thus defining a point-group series g_3^1.

All other sections of K_2 through A have as images lines which do not contain I. When A is not on S_6, these lines cut the plane curve C_6 in six variable points, and define a g_6^2. When A is on S_6, its image is at A', the intersection of the tangent at A with the line IJ.

The space sextic S_6 is defined by the quadric cone K_2 and a cubic surface F_3. The equation of the cubic surface may be written

$$w^3 + w^2 f_1(x, y, z) + wf_2(x, y, z) + f_3(x, y, z) = 0.$$

This is easily transformed to a cubic in which there is no second term. It is therefore written

$$w^3 + w(a_0x^2 + a_1xz + a_2z^2 + a_3yz + a_4y^2 + a_5xy) + b_0x^3 + b_1x^2z + b_2xz^2 + b_3z^3 + b_4yz^2 + b_5y^2z + b_6y^3 + b_7x^2y + b_8xy^2 + b_9xyz = 0.$$

By combining the above with the equation $xy = z^2$, four terms, viz., a_5xyw, b_7x^2y, b_8xy^2 and b_9xyz, can be united with other terms. The equation is thus reduced to the form

$$F_3 \equiv w^3 + w(a_0x^2 + a_1xz + a_2z^2 + a_3yz + a_4y^2) + b_0x^3 + b_1x^2z + b_2xz^2 + b_3z^3 + b_4yz^2 + b_5y^2z + b_6y^3 = 0,$$

which, taken with K_2, defines S_6.

The above projects into

$$C_6 \equiv y^3z^3 + yz\sum_{i=0}^{4} a_ix^{4-i}y^i + \sum_{i=0}^{6} b_ix^{6-i}y^i = 0,$$

an equation of a plane sextic with three branches tangent to the y-axis at I or $(0, 0, 1)$. When, however, $b_0 = 0$, the factor y can be removed and the quintic will have a tacnode at I, tangent likewise to the y-axis.

The problem now is to find all collineations which leave K_2 and F_3 invariant and to derive the corresponding plane transformations.

20. The following central involutions leave K_2 invariant:

$$E^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ y, & x, & z, & w \end{pmatrix}, \quad F^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ -y, & -x, & z, & w \end{pmatrix}, \quad G^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ x, & y, & -z, & w \end{pmatrix},$$

and a fourth,

$$\begin{pmatrix} x, & y, & z, & w \\ x, & y, & z, & -w \end{pmatrix}.$$

The last is discarded, because it can not leave F_3 invariant, except when $f_6(xy)$ vanishes identically, which happens only when the genus is less than four. These transformations project from A into

$$\begin{pmatrix} x, & y, & z \\ xy, & x^2, & yz \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ xy, & -x^2, & yz \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ -x, & y, & z \end{pmatrix}$$

respectively.

Three axial involutions are generated by the four central involutions, viz.,

$$E^2F^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ -x, & -y, & z, & w \end{pmatrix}, \quad F^2G^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ y, & x, & z, & -w \end{pmatrix}, \quad E^2G^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ y, & x, & -z, & w \end{pmatrix},$$

which project into

$$\begin{pmatrix} x, & y, & z \\ x, & -y, & z \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ xy, & x^2, & -yz \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ -xy, & x^2, & yz \end{pmatrix}.$$

The reason for using the exponent 2 with E, F, and G will appear later.

The cubic surface F_3 also is invariant under G^2 when

$$a_1 = a_3 = b_1 = b_3 = b_5 = 0,$$

and the equation is

$$w^3 + w(a_0x^2 + a_2z^2 + a_4y^2) + b_0x^3 + b_2xz^2 + b_4yz^2 + b_6y^3 = 0, \tag{1}$$

or, in the plane,

$$y^3z^3 + yz(a_0x^4 + a_2x^2y^2 + a_4y^4) + b_0x^6 + b_2x^4y^2 + b_4x^2y^4 + b_6y^6 = 0. \tag{1'}$$

The coefficients b_2 and b_4 can be absorbed by change of scale.

From the center C or $(0, 0, 1, 0)$ there are six tangents to (1), and their points of contact are on the x- and y-generators in the invariant plane $z = 0$. These project into tangents from J to the plane sextic (1′) at I and at the three other points in which the curve intersects the x-axis. There are also six bitangents from J, which make up 18, the class of the curve.

When, in F_3, $a_1 = a_3 = b_0 = b_2 = b_4 = b_6 = 0$, the equation is

$$w^3 + w(a_0x^2 + a_2z^2 + a_4y^2) + b_1x^2z + b_3z^3 + b_5y^2z = 0, \tag{2}$$

an equation invariant under the axial homology E^2F^2. It projects into the quintic

$$y^2z^3 + z(a_0x^4 + a_2x^2y^2 + a_4y^4) + b_1x^5 + b_3x^3y^2 + b_5xy^4 = 0. \tag{2'}$$

The plane quintic (2′) has an inflexional tangent at O, an ordinary tangent from O to the point of intersection with the y-axis, and six bitangents from O.

21. To obtain a four-group of the first kind, with two central and one axial involution, one may find the condition that E^2 shall leave either (1) or (2) invariant. The former yields a sextic; so the latter is chosen as being the simpler. The equation then is

$$w^3 + w\{a_0(x^2 + y^2) + a_2z^2\} + b_1z(x^2 + y^2) + b_3z^3 = 0. \tag{3}$$

The plane curve is

$$y^2z^3 + z\{a_0(x^4 + y^4) + a_2x^2y^2\} + b_1x(x^4 + y^4) + b_3x^3y^2 = 0. \qquad (3')$$

Either b_1 or b_3 can be divided out and absorbed in w.

The vertices V_1 and V_2 of the two cubic cones of perspective for (3) are on AB and project into O. The curve (3′) passes through O, and the tangent $a_0z + b_1x = 0$ is an inflexional tangent at O and an ordinary tangent on the y-axis. This tangent is the image of a double osculating plane $a_0w + b_1z = 0$, which is an inflexional tangent plane to both cubic cones.

An equation invariant under a four-group of the second kind may be obtained from (2) by finding the conditions that it be invariant under F^2G^2. The equation thus reduces to

$$w^3 + w\{a_0(x^2 + y^2) + a_2z^2\} + b_1z(x^2 - y^2) = 0. \qquad (4)$$

This gives

$$y^2z^3 + z\{a_0(x^4 + y^4) + a_2x^2y^2\} + b_1x(x^4 - y^4) = 0. \qquad (4')$$

The last coefficient reduces to unity.

This quintic has an inflexional tangent $a_0z - b_1x = 0$ which is not tangent a second time to the curve.

From (2) is also obtained an equation invariant under a diedral group of order 8. The generating operations of a cyclic G_4 which leaves K_2 invariant may be written

$$EF \equiv \begin{pmatrix} x, & y, & z, & w \\ -x, & y, & -iz, & iw \end{pmatrix},$$

which projects into

$$\begin{pmatrix} x, & y, & z \\ x, & iy, & -z \end{pmatrix}.$$

Equation (2), if invariant under EF, reduces to

$$w^3 + a_2wz^2 + z(b_1x^2 + b_5y^2) = 0, \qquad (5)$$

which is manifestly invariant also under E^2 and F^2. This projects into the quintic

$$y^2z^3 + a_2x^2y^2z + x(b_1x^4 + b_5y^4) = 0. \qquad (5')$$

The last two coefficients can be changed to unity.

To each of the five preceding groups there is a corresponding group in § 1.

22. There is also a cyclic group of order 4, whose collineation of period 2 is the central involution G^2. Its generating operation is

$$G \equiv \begin{pmatrix} x, & y, & z, & w \\ -x, & y, & iz, & w \end{pmatrix},$$

which projects into

$$\begin{pmatrix} x, & y, & z \\ ix, & y, & z \end{pmatrix}.$$

This differs essentially from the last group, which contains an axial involution. The equations belonging to this group are

$$w^3 + w(a_0x^2 + a_4y^2) + b_2xz^2 + b_6y^3 = 0 \tag{6}$$

and

$$y^2z^3 + z(a_0x^4 + a_4y^4) + b_2x^4y + b_6y^5 = 0. \tag{6'}$$

The coefficients a_0 and a_4 can be absorbed. There is no corresponding equation in § 1.

The center of the homology is at J. The tangent $a_0z + b_2y = 0$ has contact of the fourth order. This counts as five tangents and JI counts as two. There are, moreover, from J, three tangents to points on the x-axis, each with contact of the third order.

23. The transformations

$$H_y \equiv \begin{pmatrix} x, & y, & z, & w \\ \omega^2x, & \omega y, & z, & w \end{pmatrix},$$

where $\omega^3 = 1$, form a cyclic G_3. Here every point of the polar line $x = y = 0$ is invariant. The plane transformations are of the form

$$\begin{pmatrix} x, & y, & z \\ x, & \omega y, & z \end{pmatrix}.$$

The subscript y distinguishes this transformation from one following, where the axis of homology is the x-axis. The equations are

$$w^3 + a_2z^2w + b_0x^3 + b_1y^3 + b_2z^3 = 0 \tag{7}$$

and

$$y^3z^3 + a_2x^2y^3z + b_0x^6 + b_1y^6 + b_2x^3y^3 = 0. \tag{7'}$$

The three central involutions

$$K \equiv \begin{pmatrix} x, & y, & z, & w \\ b_1^{2/3}\omega y, & \omega^2 b_0^{2/3}x, & b_0^{1/3}b_1^{1/3}z, & b_0^{1/3}b_1^{1/3}w \end{pmatrix},$$

respectively

$$\begin{pmatrix} x, & y, & z \\ b_0^{1/3}b_1^{1/3}xy, & \omega b_0^{2/3}x^2, & b_0^{1/3}b_1^{1/3}yz \end{pmatrix},$$

leave the curves invariant. Hence they belong to a diedral G_6. The three centers of perspective V_1, V_2, V_3 lie on AB and project into O. By a change of scale the coefficients of x^3 and y^3 can be reduced to unity. This simplifies the last transformations. The planes of perspective then become $x - \omega y = 0$, which project into the lines $x^2 - \omega y^2 = 0$.

This curve has six tritangents from O, as there are six planes through AB tangent to all three cones of perspective. The sextic (6′), § 1, corresponds to this one.

When, in (7), $b_2 = 0$, the central involution G^2 leaves the curve invariant and the group is a diedral group of order 12. The equations are

$$w^3 + a_2 z^2 w + x^3 + y^3 = 0 \tag{8}$$

and

$$y^3 z^3 + a_2 x^2 y^3 z + x^6 + y^6 = 0. \tag{8'}$$

24. There is also a cyclic G_3 which leaves invariant all points of the x-generator. Its transformations are

$$H_x \equiv \begin{pmatrix} x, & y, & z, & w \\ \omega^2 x, & \omega y, & z, & \omega w \end{pmatrix},$$

which project into

$$\begin{pmatrix} x, & y, & z \\ \omega x, & y, & z \end{pmatrix}.$$

The equations are

$$w^3 + w(a_1 xz + a_4 y^2) + b_0 x^3 + b_6 y^3 + b_3 z^3 = 0 \tag{9}$$

and

$$y^3 z^3 + y^2 z(a_1 x^3 + a_4 y^3) + b_0 x^6 + b_6 y^6 + b_3 x^3 y^3 = 0. \tag{9'}$$

The coefficients b_0 and b_6 can be absorbed by change of scale.

Attention has been called to differences in the space transformations H_x and H_y. The corresponding plane homologies differ merely in respect to the curve; the one has a secant through the singular point as axis of homology, while the other has the tangent at that point as axis.

If, in equation (9′), the function $f_6(x, y)$ be the sextic covariant of $f_4(x, y)$, the equations, after absorbing coefficients, may be written

$$w^3 + aw(xz + y^2) + x^3 + 20z^3 - 8y^3 = 0 \tag{10}$$

and

$$y^3 z^3 + ay^2 z(x^3 + y^3) + x^6 + 20x^3 y^3 - 8y^6 = 0. \tag{10'}$$

The space and plane sextics are left invariant by the four-group whose respective transformations are the following:

$$\begin{pmatrix} x, & y, & z, & w \\ x+4y-4z, & x+y+2z, & -x+2y+z, & -3w \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ (-x+2y)(x+y), & (x+y)^2, & -3yz \end{pmatrix},$$

$$\begin{pmatrix} x, & y, & z, & w \\ \omega^2 x+4y-4\omega z, & \omega x+\omega^2 y+2z, & -x+2\omega y+\omega^2 z, & -3\omega^2 w \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ (-\omega x+2y)(\omega^2 x+y), & (\omega^2 x+\omega y)^2, & -3\omega^2 yz \end{pmatrix},$$

$$\begin{pmatrix} x, & y, & z, & w \\ \omega x+4y-4\omega^2 z, & x+\omega^2 y+2\omega z, & -\omega^2 x+2\omega y+z, & -3\omega^2 w \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ (2y-\omega^2 x)(\omega y+x), & (\omega y+x)^2, & -3\omega^2 yz \end{pmatrix}.$$

The group is therefore the tetraedron group generated by the above four-group and the preceding group of order 3.

A cyclic G_6 can be obtained from the transformations H_x and G^2. Their product gives

$$\begin{pmatrix} x, & y, & z, & w \\ \omega^2 x, & \omega y, & -z, & w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ -\omega x, & y, & z \end{pmatrix};$$

and the equations are

$$w^3 + a_4 y^2 w + b_0 x^3 + b_6 y^3 = 0 \tag{11}$$

and

$$y^3 z^3 + a_4 y^5 z + b_0 x^6 + b_6 y^6 = 0, \tag{11'}$$

where b_0 and b_6 can be absorbed. All points of the x-generator are invariant.

A cyclic G_{12} is obtained whose transformations may be represented by

$$G^3 E^2 F^2 H_x \equiv \begin{pmatrix} x, & y, & z, & w \\ -\omega^2 x, & y, & i\omega z, & -w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ \omega x, & -iy, & iz \end{pmatrix}.$$

The equations for this group are

$$w^3 + a_4 y^2 w + b_0 x^3 = 0 \tag{12}$$

and

$$y^3 z^3 + a_4 y^5 z + b_0 x^6 = 0; \tag{12'}$$

and both coefficients can be absorbed by change of scale.

25. For the cyclic G_5 we may have

$$L \equiv \begin{pmatrix} x, & y, & z, & w \\ \theta^2 x, & y, & \theta z, & w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ \theta x, & y, & z \end{pmatrix},$$

where $\theta^5 = 1$; and the equations are

$$w^3 + a_4 y^2 w + b_1 x^2 z + b_6 y^3 = 0 \tag{13}$$

and

$$y^2 z^3 + a_4 y^4 z + b_1 x^5 + b_6 y^5 = 0. \tag{13'}$$

The coefficient b_1 can be dropped.

This is a subgroup of the cyclic G_{10} whose operations are

$$LE^2F^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ -\theta^2 x, & -y, & \theta z, & w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ \theta x, & -y, & z \end{pmatrix},$$

with the equations

$$w^3 + a_4 y^2 w + b_1 x^2 z = 0; \tag{14}$$

$$y^2 z^3 + a_4 y^4 z + b_1 x^5 = 0. \tag{14'}$$

Here both coefficients can be changed to unity.

26. In the following cases $f_4(x, y)$ vanishes identically. There will always be a central collineation of period 3, viz.,

$$H_z \equiv H_x H_y \equiv \begin{pmatrix} x, & y, & z, & w \\ x, & y, & z, & \omega w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ x, & y, & \omega z \end{pmatrix}.$$

The equations are

$$w^3 + b_0 x^3 + b_1 x^2 z + b_2 x z^2 + b_3 z^3 + b_4 y z^2 + b_5 y^2 z + b_6 y^3 = 0 \tag{15}$$

and

$$y^3 z^3 + \sum_{i=0}^{6} b_i x^{6-i} y^i = 0. \tag{15'}$$

The following equations, with their transformations, arise from special values of the coefficients b_i:

$$w^3 + b_0 x^3 + b_2 x z^2 + b_4 y z^2 + b_6 y^3 = 0, \tag{16}$$

$$y^3 z^3 + b_0 x^6 + b_2 x^4 y^2 + b_4 x^2 y^4 + b_6 y^6 = 0, \tag{16'}$$

whose transformations are G^2 respectively G'^2, which with H_z respectively H'_z form a cyclic G_6. The accents as before are used to denote the plane transformations. The coefficients b_0 and b_6 can be absorbed.

The curve

$$w^3 + b_1 x^2 z + b_3 z^3 + b_5 y^2 z = 0 \tag{17}$$

has the axial involution E^2F^2. It projects into

$$y^2 z^3 + b_1 x^5 + b_3 x^3 y^2 + b_5 x y^4 = 0. \tag{17'}$$

Now as b_1 and b_2 can be absorbed, the transformation E^2 belongs to the group. The group is of order 12, with a four-group and three cyclic groups of order 6 as subgroups.

If, in (16), $b_2 = b_4$ and $b_0 = b_6$, we have

$$w^3 + x^3 + y^3 + b_2 z^2(x + y) = 0, \tag{16a}$$

with the transformations E, G and K. This sextic belongs to a G_{12} of the same kind as that for (17). It projects into a sextic.

There is also

$$w^3 + b_0 x^3 + b_3 z^3 + b_6 y^3 = 0, \tag{18}$$

with the transformations H_x, H_y and H_z. It projects into

$$y^3 z^3 + b_0 x^6 + b_3 x^3 y^3 + b_6 y^6 = 0. \tag{18'}$$

Since b_0 and b_6 are reducible to unity, there is also the central perspectivity E^2. Thus a group of order 18 is generated.

When $b_3 = 0$, the equation

$$w^3 + b(x^3 + y^3) = 0 \tag{19}$$

is invariant under the perspectivity G^2. It projects into

$$y^3 z^3 + b(x^6 + y^6) = 0. \tag{19'}$$

The group is a G_{36} with the above G_{18} as invariant subgroup. We can absorb b.

When, in (17), $b_3 = 0$, we have

$$w^3 + z(x^2 + y^2) = 0, \tag{20}$$

$$y^2 z^3 + x(x^4 + y^4) = 0, \tag{20'}$$

with the transformations EF, respectively $E'F'$. These equations belong to a G_{72} with the G_{18} as invariant subgroup.

Last of all there are the equations

$$w^3 + b_1 x^2 z + b_6 y^3 = 0, \tag{21}$$

$$y^2 z^3 + b_1 x^5 + b_6 y^5 = 0, \tag{21'}$$

where b_1 and b_6 can be omitted. These curves belong to a cyclic G_{15} generated by H_z, L and H'_z, L' respectively.

27. We have seen that space sextics of genus 4 may be projected into plane sextics with six distinct double points if the center of projection be selected without the hyperboloid or cone. The double points moreover will lie on a

conic.* Thus if three of the double points are collinear, the remaining three must be either collinear or coincident.

Any C_6 with six distinct double points can be transformed by aid of adjoint cubics into a quintic with two nodes either distinct or coincident; thence, by projecting from the corresponding space sextic, into a C_6 whose six double points lie on a conic.

The plane sextic with two distinct triple points may be inverted into a binodal quintic if the triangle of inversion be chosen with center O at an ordinary point of the curve and the vertices I and J at the triple points. The equation of the quintic then is

$$x^3\phi_2(y, z) + x^2\psi_3(y, z) + xyf_3(y, z) + y^2\phi_3(y, z) = 0,$$

with nodes at $(1, 0, 0)$ and $(0, 0, 1)$.

This may be transformed to a new reference triangle with vertex O' at a node, and I' and J' ordinary points of the quintic collinear with the second node. By inversion a sextic is obtained with two double points and a triple point having two of its branches tangent. The triple point thus counts as four double points. The transformation from the original sextic is of order 3.

The binodal quintic may likewise be inverted into a sextic with a triple point and three non-collinear double points by selecting the triangle of reference with O' at a node and I' and J' at ordinary points on the quintic not collinear with a node. Two of these double points unite in a tacnode when one of the vertices I' or J' is at the point where the quintic intersects the line which joins the nodes.

The plane sextic with two coincident triple points may be transformed into a quintic with a tacnode by using the quadric transformation with OIJ coincident at the tacnode and a base conic tangent to the tacnodal tangent. The arbitrary constant of the quadric transformation is so chosen that the transformed curve will be a quintic.

Finally, if the triangle of inversion be selected with O at the tacnode and I and J ordinary points of the curve, such that neither OI nor OJ will be the tacnodal tangent, the inverted curve is the sextic† with a triple point at O and three double points on IJ.

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* Halphen: Mémoire sur la Classification des Courbes gauches algébriques, *Journal de l'École Polytechnique*, cahier LII (1882), pp. 1–200.

† L. Kraus: Note über aussergewöhnliche Specialgruppen auf algebraischen Curven, *Math. Ann.*, Vol. XVI (1879).

A sextic with a quadruple point possesses a g_2^1, is therefore hyperelliptic and belongs to the following section.

§ 3. *The Hyperelliptic Curves.*

28. As was noted at the beginning of this paper, the hyperelliptic C_m (4) can not be transformed to a C_5. For when C_m is hyperelliptic, $m \geqq p+2$, or, in this case, $m \geqq 6$. Reducing the equation to the canonical form $y^2 = f_{2p+2}(x)$ gives $y^2 = f_{10}(x)$, or, in homogeneous coördinates,

$$y^2z^8 = \sum_{n=0}^{10} a_n x^{10-n} z^n, \tag{1}$$

which admits the transformation

$$T \equiv \begin{pmatrix} x, & y, & z \\ x, & -y, & z \end{pmatrix}.$$

If the coefficients of the odd powers of x all vanish, the equation

$$y^2z^8 = a_0x^{10} + a_2x^8z^2 + a_4x^6z^4 + a_6x^4z^6 + a_8x^2z^8 + a_{10}z^{10} \tag{2}$$

belongs to the four-group whose operations are

$$\begin{pmatrix} x, & y, & z \\ \pm x, & \pm y, & z \end{pmatrix}.$$

When $a_0 = a_{10}$, $a_2 = a_8$ and $a_4 = a_6$, the curve

$$z^8y^2 = a_0(x^{10} + z^{10}) + a_2(x^8z^2 + x^2z^8) + a_4(x^6z^4 + x^4z^6) \tag{3}$$

possesses, in addition to the above four-group, the transformation

$$R \equiv \begin{pmatrix} x, & y, & z \\ x^4z, & yz^4, & x^5 \end{pmatrix},$$

an operation of the second order commutative with the other operations, and the equation belongs therefore to a diedral G_8.

When, in (3), $a_2 = a_1 = 0$, the equation

$$y^2z^8 = a_0(x^{10} + z^{10}) \tag{4}$$

possesses, in addition to the above transformation of G_8, a cyclic G_{10} whose operations are

$$S \equiv \begin{pmatrix} x, & y, & z \\ \theta x, & y, & z \end{pmatrix},$$

where $\theta^{10} = 1$, and the group of the equation is of order 40. For consider the generating operations R, S and T. There are ten operations of the form SR,

including R itself. There are nine more of the form RS, but they are the same as those of the form SRT and must not be counted twice. There are ten of the form S, including identity. These twenty operations, combined with T, form the G_{40}.

Returning to equation (1), it may be that $a_n = a_{10-n}$ for $n = 1, 2, \ldots, 10$. The equation is then of the form

$$y^2z^8 = a_0(x^{10} + z^{10}) + a_1xz(x^8 + z^8) + a_2x^2z^2(x^6 + z^6) + a_3x^3z^3(x^4 + z^4) \\ + a_4x^4z^4(x^2 + z^2) + a_5x^5z^5, \qquad (5)$$

which is found to possess the transformations T and R of the above group, but not S. This equation, like (2), belongs to a four-group, and it reduces to (3) with the diedral G_8 when $a_1 = a_3 = 0$.

If $a_1 = a_2 = a_3 = a_4$ in equation (5), we have

$$y^2z^8 = a_0(x^{10} + z^{10}) + a_5x^5z^5, \qquad (6)$$

which possesses the subgroup, a cyclic G_5 with generating operations

$$Q \equiv \begin{pmatrix} x, & y, & z \\ \theta x, & y, & z \end{pmatrix},$$

where $\theta^5 = 1$. This equation has therefore five transformations Q_iR, including R itself, five of form Q, and combining with T a group G_{20} is found, a subgroup of the G_{40} in equation (4). While R and Q are not commutative, it is clear that $Q_iR = RQ_j$.

If the equation had the form of (6), except that the coefficients of x^{10} and z^{10} are unequal, then

$$y^2z^8 = a_0x^{10} + a_5x^5z^5 + a_{10}z^{10} \qquad (7)$$

has the transformations Q and T, but not R. Hence the group is a cyclic G_{10}.

If the curve be reduced to the form

$$y^2 = x^{2p+1} + 1 \text{ or } y^2z^7 = x^9 + z^9, \qquad (8)$$

the curve has

$$\begin{pmatrix} x, & y, & z \\ mx, & \pm y, & z \end{pmatrix},$$

where $m^9 = 1$. These are operations of a cyclic G_{18}.

When the equation is of the form

$$y^2 = x(x^{2p} + 1) \text{ or } y^2z^7 = x(x^8 + z^8), \tag{9}$$

it has the transformation R, and also

$$V \equiv \begin{pmatrix} x, & y, & z \\ n^2x, & ny, & z \end{pmatrix},$$

where $n^{16} = 1$. The operation V is of period 16. The operations V and R are not commutative, but any product RV is one of the products VR, and the square of either is

$$\begin{pmatrix} x, & y, & z \\ x, & n^8y, & z \end{pmatrix} \equiv V^8,$$

which is identity or T, according to which one of the sixteenth roots of unity n is. The operations are V^i and RV^i, where $i = 0, 1, \ldots, 15$. The group is a G_{32}, neither diedral nor cyclic.

In closing, let me state that in certain cases details have been given which might have been omitted had Wiman's paper been more easily accessible to the reader.

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